Modern approaches to quantum gravity

Solution 9 Fall 2025

1. Basics of 2D CFT

(a) By symmetry, we can assume that all indices are ordered, so there are at most $\ell+1$ components $M_{11\ldots 1}$, $M_{11\ldots 12}$, ..., $M_{2\ldots 2}$. Tracelessness implies that

$$\delta_{\mu\nu} M_{\mu\nu\alpha\ldots\beta} = M_{11\alpha\ldots\beta} + M_{22\alpha\ldots\beta} = 0 \tag{1}$$

hence we can eliminate any pair of (11) indices in favor of a pair of (22) indices. Thus any tensor with spin $\ell \geq 2$ has 2 independent components: $M_{122\ldots 2}$ and $M_{2\ldots 2}$ (the exception is $\ell = 0$, which has a single component).

We still need to show that $M_{z\ldots z} := M$ and $M_{\bar{z}\ldots\bar{z}} := \bar{M}$ are independent. In fact, we can simply show that all other components vanish. This is again due to tracelessness. In the z, $\bar{z}$ coordinates, the flat-space metric reads $ds^2 = dzd\bar{z}$, so the components of the metric are

$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}, \quad g_{zz} = g_{\bar{z}\bar{z}} = 0 \tag{2}$$

and the inverse metric satisfies $g^{zz} = g^{\bar{z}\bar{z}} = 0$, $g^{z\bar{z}} = g^{\bar{z}z} = 2$. Thus

$$0 = g^{\mu\nu} M_{\mu\nu\alpha\ldots\beta} = 4 M_{z\bar{z}\alpha\ldots\beta} \tag{3}$$

for any indices $\alpha, \ldots, \beta$. In conclusion, the only non-zero components of any traceless symmetric tensor are M and $\bar{M}$. Finally, using the Jacobian, we find that

$$M_{12\ldots 2} = i^{\ell-1}\left(M + (-1)^{\ell-1}\bar{M}\right) \quad \text{and} \quad M_{2\ldots 2} = i^{\ell}\left(M + (-1)^{\ell}\bar{M}\right). \tag{4}$$

Conservation of a tensor means that

$$0 = g^{\mu\nu}\partial_\mu M_{\nu\alpha\ldots\beta} = 2\left(\partial\bar{M}_{\alpha\ldots\beta} + \bar{\partial} M_{\alpha\ldots\beta}\right) \tag{5}$$

using the shorthand notation $\partial = \partial/\partial z$, $\bar{\partial} = \partial/\partial\bar{z}$. In particular, by setting all coordinates either equal to z or to $\bar{z}$, we find that

$$\bar{\partial} M = 0, \quad \partial\bar{M} = 0. \tag{6}$$

In other words, the component M depends only on z, and $\bar{M}$ only depends on $\bar{z}$.

Under a finite rotation R, a tensor transforms as

$$T^{\mu\ldots\nu}(x) \mapsto R^{\mu'}_{\mu} \ldots R^{\nu'}_{\nu} T^{\mu'\ldots\nu'}(x'). \tag{7}$$

In the z, $\bar{z}$ coordinates, a rotation by an angle θ can be represented as

$$R^{\mu}_{\nu} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}. \tag{8}$$

Thus

$$M(z,\bar{z}) \mapsto e^{i\ell\theta} M(z',\bar{z}'), \quad \bar{M}(z,\bar{z}) \mapsto e^{-i\ell\theta} \bar{M}(z',\bar{z}'), \tag{9}$$

with $z' = e^{i\theta} z$, $\bar{z}' = e^{-i\theta}\bar{z}$.

Finally, in the z, $\bar{z}$ coordinates, parity ($y \mapsto -y$) acts as

$$P^{\mu}_{\nu} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{10}$$

so that (up to an intrinsic parity $\eta_M = \pm 1$),

$$M(z,\bar{z}) \mapsto \bar{M}(\bar{z},z), \quad \bar{M}(z,\bar{z}) \mapsto M(\bar{z},z). \tag{11}$$

(b) We can start from the known 2-pt function

$$\langle J_\mu(x) J_\nu(y)\rangle = k_J \frac{I_{\mu\nu}(x-y)}{|x-y|^{2\Delta}} \tag{12}$$

for some constant $k_J > 0$, where for conserved currents, obviously $\Delta = d - 1 = 1$. We can get the component correlators using

$$J = (\partial x^\mu) J_\mu = \frac{1}{2}(J_1 - iJ_2), \quad \bar{J} = (\bar{\partial} x^\mu) J_\mu = \frac{1}{2}(J_1 + iJ_2). \tag{13}$$

Then we find that

$$\langle J(z,\bar{z}) J(w,\bar{w})\rangle = -\frac{k_J}{2} \frac{1}{(z-w)^2}. \tag{14}$$

Setting $\Delta = 1$, we find in particular that

$$\langle J(z,\bar{z}) J(w,\bar{w})\rangle = -\frac{k_J}{2(z-w)^2} \tag{15}$$

so we confirm that the correlator only depends on the holomorphic coordinates z, w. Likewise

$$\langle \bar{J}(z,\bar{z}) \bar{J}(w,\bar{w})\rangle = -\frac{k_J}{2(\bar{z}-\bar{w})^2}, \quad \langle J(z,\bar{z}) \bar{J}(w,\bar{w})\rangle = 0. \tag{16}$$

Parity is automatically preserved in this way. Conversely, without parity invariance we could write the same 2-pt functions with different constants k_J in the $\langle JJ\rangle$ and $\langle \bar{J}\bar{J}\rangle$ correlators, and such correlators would be conformally invariant.

Likewise,

$$\langle \bar{J}(z,\bar{z}) \bar{J}(w,\bar{w})\rangle = -\frac{k_J}{2} \frac{1}{(\bar{z}-\bar{w})^2}, \quad \langle J(z,\bar{z}) \bar{J}(w,\bar{w})\rangle = 0. \tag{17}$$

For the stress-energy tensor, we obtain

$$\langle T(z) T(w)\rangle = \frac{c/2}{(z-w)^4}, \quad \langle T(z) \bar{T}(w)\rangle = 0, \quad \langle \bar{T}(z) \bar{T}(w)\rangle = \frac{\bar{c}/2}{(\bar{z}-\bar{w})^4}. \tag{18}$$

(c) It is trivial to show that the modes with labels $m, n \in \{-1, 0, 1\}$ form a subalgebra. In particular, their commutator only gives modes in $\{-1, 0, 1\}$. It is a standard fact from complex analysis that the only conformal transformations of the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ are

$$f(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0. \tag{19}$$

The group of these transformations is called the Möbius or global conformal group, denoted by $\mathrm{PSL}(2, \mathbb{C})$. It is generated by translations, rotations, dilations, and inversions:

$$f_{\mathrm{tr}}(z) = z + a, \quad f_{\mathrm{rot}}(z) = e^{i\theta} z, \quad f_{\mathrm{dil}}(z) = cz, \quad f_{\mathrm{inv}}(z) = \frac{1}{z}. \tag{20}$$

The global conformal group enlarges the group of rigid transformations of $\mathbb{C}$ (that is, translations and rotations) by adding scale transformations and mappings that turn the complex plane "inside out." An interesting fact is that $\mathrm{PSL}(2, \mathbb{C}) \cong \mathrm{SO}(3, 1)$.

The global Virasoro generators have physical interpretations:

- $L_0 + \bar{L}_0$ generates dilatations and is the Hamiltonian.
- $L_0 - \bar{L}_0$ generates rotations and is the angular momentum.
- L_{-1} and $\bar{L}_{-1}$ generate translations.
- L_1 and $\bar{L}_1$ generate special conformal transformations.

(d) Since the generator of translation is the energy-momentum tensor, we have

$$H = L_0 + \bar{L}_0, \tag{21}$$

which means that h and $\bar{h}$ are the energies of the states.

Let $|\psi\rangle \in \mathcal{H}_{\mathrm{CFT}}$ be an eigenvector of L_0 with weight h. Using the Virasoro algebra, we find that the state $L_n|\psi\rangle$ is also an eigenvector of L_0 with eigenvalue shifted by n:

$$L_0(L_n|\psi\rangle) = ([L_0, L_n] + L_n L_0)|\psi\rangle = (h - n)L_n|\psi\rangle. \tag{22}$$

Since L_n lowers the energy of a state by n, and the Hamiltonian is bounded from below, there must exist an L_0 eigenstate $|\psi\rangle$ that is annihilated by L_n for all $n \geq 0$. Any state $|\psi\rangle \in \mathcal{H}_{\mathrm{CFT}}$ for which

$$L_0|\psi\rangle = h|\psi\rangle, \quad L_n|\psi\rangle = 0 \quad \text{for all } n > 0 \tag{23}$$

is called a primary or highest weight state.

(e) At level 1, there is a single state $L_{-1}|O\rangle$ with norm

$$\langle O|L_1 L_{-1}|O\rangle = 2\langle O|L_0|O\rangle = 2h\langle O|O\rangle = 2h, \tag{24}$$

where we normalized the state to have norm 1. This implies that for unitarity, we need $h > 0$.

At level 2, there are two possible states:

$$|\psi_{1,1}\rangle = L_{-1}^2|O\rangle, \quad |\psi_2\rangle = L_{-2}|O\rangle. \tag{25}$$

The 2×2 Gram matrix at level 2 is

$$M^{(2)}(c,h) = \begin{pmatrix} \langle \psi_{1,1}|\psi_{1,1}\rangle & \langle \psi_{1,1}|\psi_2\rangle \\ \langle \psi_2|\psi_{1,1}\rangle & \langle \psi_2|\psi_2\rangle \end{pmatrix} = \begin{pmatrix} 4h(2h+1) & 6h \\ 6h & \frac{c}{2}+4h \end{pmatrix}. \tag{26}$$

At level 3, we have

$$|\psi_1\rangle = L_{-1}|O\rangle, \tag{27}$$

with the 3×3 Gram matrix

$$M^{(3)}(c,h) = \begin{pmatrix} \langle \psi_1|\psi_1\rangle & \langle \psi_1|\psi_{1,1}\rangle & \langle \psi_1|\psi_2\rangle \\ \langle \psi_{1,1}|\psi_1\rangle & \langle \psi_{1,1}|\psi_{1,1}\rangle & \langle \psi_{1,1}|\psi_2\rangle \\ \langle \psi_2|\psi_1\rangle & \langle \psi_2|\psi_{1,1}\rangle & \langle \psi_2|\psi_2\rangle \end{pmatrix} = \begin{pmatrix} 2h & 0 & 0 \\ 0 & 4h(2h+1) & 6h \\ 0 & 6h & \frac{1}{2}(c+8h) \end{pmatrix}. \tag{28}$$

This matrix should be positive definite. One can solve for its eigenvalues. You can read about it in Di Francesco page 207. The trace is

$$\mathrm{Tr}\left(M^{(3)}(c,h)\right) = \frac{c}{2} + 2h(5+4h) \to \frac{c}{2} \text{ when } h = 0 \tag{29}$$

and the determinant

$$\det\left(M^{(3)}(c,h)\right) = 4h(c + 2h(c+8h-5)) \to 0 \text{ when } h = 0. \tag{30}$$

The eigenvalues, with $h = 0$, are 0, 0, and $c/2$, such that we get $c > 0$ again. Null states are states for which norms are zero.

2. OPE and free scalars

(a) We can prove this result using the sampling property of the delta function. Let R be a closed domain in the complex plane. Then the divergence theorem in complex coordinates says that

$$\int_R d^2z(\partial v_z + \overline{\partial} v_{\overline{z}}) = i\oint_{\partial R}(v_z d\overline{z} - v_{\overline{z}} dz), \tag{31}$$

where v_α is a vector field, and the contour ∂R is traversed anticlockwise.

Now let $f(z)$ be a holomorphic test function and suppose that the region R encloses the origin.

$$\int_R d^2z\, \partial\overline{\partial} \ln|z|^2 f(z) = \int_R d^2z\, \overline{\partial}\left(\frac{1}{z}f(z)\right) = -i\oint_{\partial R}\frac{dz}{z}f(z) = 2\pi f(0), \tag{32}$$

where we use the residue theorem in the final line.

Similarly, for an antiholomorphic test function $f(\overline{z})$,

$$\int_R d^2z\, \partial\overline{\partial} \ln|z|^2 f(\overline{z}) = i\oint_{\partial R}\frac{d\overline{z}}{\overline{z}}f(\overline{z}) = 2\pi f(0). \tag{33}$$

This means that

$$\partial\overline{\partial}\ln|z|^2 = 2\pi\delta(z,\overline{z}), \tag{34}$$

by the definition of the delta function.

(b) The free field $X^\mu(z,\bar{z})$ satisfies the OPE

$$X^\mu(z,\bar{z})X^\nu(w,\bar{w}) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-w|^2. \tag{35}$$

Taking derivatives, this implies

$$\partial X^\mu(z)\partial X^\nu(w) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\frac{1}{(z-w)^2}. \tag{36}$$

We can invert the mode expansion $\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\alpha_n^\mu z^{-n-1}$ via

$$i\sqrt{\frac{2}{\alpha'}}\oint\frac{dz}{2\pi i}z^n\partial X^\mu = \oint\frac{dz}{2\pi i}\sum_m \alpha_m^\mu z^{n-m-1} = \sum_m \alpha_m^\mu\delta_{m,n} = \alpha_n^\mu. \tag{37}$$

Then, we can use the OPE (which has implicit radial ordering) to find

$$[\alpha_m^\mu,\alpha_n^\nu] = -\frac{2}{\alpha'}\oint_{w=0}\frac{dw}{2\pi i}w^n\oint_{z=w}\frac{dz}{2\pi i}z^m R(\partial X^\mu(z)\partial X^\nu(w)), \tag{38}$$

$$= \oint_{w=0}\frac{dw}{2\pi i}w^n\oint_{z=w}\frac{dz}{2\pi i}\eta^{\mu\nu}\frac{z^m}{(z-w)^2}, \tag{39}$$

$$= \oint_{w=0}\frac{dw}{2\pi i}w^n\oint_{z=w}\frac{dz}{2\pi i}\eta^{\mu\nu}\frac{mz^{m-1}}{z-w}, \tag{40}$$

$$= \oint_{w=0}\frac{dw}{2\pi i}mw^{m+n-1}\eta^{\mu\nu}, \tag{41}$$

$$= m\eta^{\mu\nu}\delta_{m+n,0}, \tag{42}$$

where we integrated by parts in the third line.

(c) The holomorphic stress tensor is

$$T(z) = -\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu : (z),$$

and we will need the OPE

$$\partial X^\mu(z)X^\nu(w) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\frac{1}{z-w}.$$

Then we compute

$$T(z)X^\mu(w) = -\frac{1}{\alpha'} : \partial X^\nu\partial X_\nu(z) : X^\mu(w) \tag{43}$$

$$= -\frac{2}{\alpha'} : \partial X^\nu[\partial X_\nu(z) : X^\mu(w)] + \dots \tag{44}$$

$$= \frac{\partial X^\mu(z)}{z-w} + \dots = \frac{\partial X^\mu(w)}{z-w} + \dots \tag{45}$$

where the [...] denotes a resolution of the OPE.
To find the OPE with $\partial^n X^\mu(w)$ we simply differentiate n times with respect to w. By the Leibniz rule,

$$T(z)\partial^n X^\mu(w) = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\partial^{k+1}X^\mu(w)\partial_w^{n-k}\left(\frac{1}{z-w}\right) + \dots \tag{46}$$

$$= \sum_{k=0}^{n}\frac{n!}{k!}\frac{\partial^{k+1}X^\mu(w)}{(z-w)^{n-k+1}} + \dots \tag{47}$$

The second-order pole is at $k = n-1$ and has coefficient $n\partial^n X^\mu(w)$, implying the conformal weight $h = n$. Since $\partial^n X^\mu(w)$ is holomorphic, it is also clear that $\bar{h} = 0$. However, when $n > 1$, the operator is not primary since it contains poles of order greater than 2.

3. Weyl transformations are anomalous in $d = 2$

(a) The Ricci scalar of the metric $ds^2 = e^{2\sigma(z,\bar{z})}dzd\bar{z}$ is

$$R = -8e^{-2\sigma}\partial_z\partial_{\bar{z}}\sigma \,. \tag{48}$$

Thus the Weyl anomaly condition gives

$$g^{\mu\nu}\langle T_{\mu\nu}\rangle = 2g^{z\bar{z}}\langle T_{z\bar{z}}\rangle = \frac{c}{24\pi}\left(-8e^{-2\sigma}\partial_z\partial_{\bar{z}}\sigma\right) \,, \tag{49}$$

which implies

$$\langle T_{z\bar{z}}\rangle = -\frac{c}{12\pi}\partial_z\partial_{\bar{z}}\sigma \,. \tag{50}$$

The conservation can be written as

$$g^{\mu\rho}\nabla_\rho\langle T_{\mu\nu}\rangle = 0 \tag{51}$$

To obtain an equation for $\langle T_{zz}\rangle$, choose $\nu = z$. This gives

$$g^{\bar{z}z}\nabla_z\langle T_{\bar{z}z}\rangle + g^{z\bar{z}}\nabla_{\bar{z}}\langle T_{zz}\rangle = 0 \tag{52}$$

The only non-vanishing Christofell symbols are

$$\Gamma^z_{zz} = 2\partial_z\sigma \qquad \Gamma^{\bar{z}}_{\bar{z}\bar{z}} = 2\partial_{\bar{z}}\sigma \,. \tag{53}$$

The conservation thus gives

$$\partial_{\bar{z}}\langle T_{zz}\rangle = -\partial_z\langle T_{z\bar{z}}\rangle + 2(\partial_z\sigma)\langle T_{z\bar{z}}\rangle = \frac{c}{12\pi}\partial_{\bar{z}}(\partial_z^2\sigma - (\partial_z\sigma)^2) \tag{54}$$

This gives the desired form

$$\langle T_{zz}\rangle = \frac{c}{12\pi}(\partial_z^2\sigma - (\partial_z\sigma)^2) \,. \tag{55}$$

(b) The metric of a Euclidean cylinder $M = \mathbb{R}\times S^1$ (with radius $R = 1$) is given by:

$$ds^2_{\text{cyl}} = d\tau^2 + d\phi^2, \quad \tau\in\mathbb{R},\, \phi\sim\phi+2\pi.$$

Consider the map from flat space to M:

$$z\mapsto e^{\tau+i\phi}, \quad \bar{z}\mapsto e^{\tau-i\phi}.$$

The metrics relate as:

$$ds^2_{\text{flat}} = dzd\bar{z} = (z\bar{z})(d\tau+id\phi)(d\tau-id\phi) = e^{2\tau}ds^2_{\text{cyl}}.$$

Thus, $(g_{\text{cyl}})_{\mu\nu} = e^{2\sigma}\delta_{\mu\nu}$ with $\sigma = -\tau$. In $z, \bar{z}$ coordinates, this gives

$$\sigma = -\frac{1}{2}(\ln z + \ln \bar{z}) \tag{56}$$

Thus,

$$\langle T_{zz} \rangle_{\text{cyl}} = \frac{c}{12\pi} \frac{1}{4z^2} \tag{57}$$

This is related to τ, σ coordinates by

$$\langle T_{zz} \rangle_{\text{cyl}} = \frac{1}{4z^2}(\langle T_{\tau\tau} \rangle_{\text{cyl}} - \langle T_{\sigma\sigma} \rangle_{\text{cyl}}) \tag{58}$$

The Weyl anomaly predicts:

$$\langle T^{\mu}_{\mu} \rangle_M = \frac{c}{24\pi} R[M], \tag{59}$$

where $R[M]$ is the Ricci scalar. Since $R[M] = 0$ for the cylinder, it follows:

$$\langle T_{\tau\tau} \rangle_{\text{cyl}} + \langle T_{\phi\phi} \rangle_{\text{cyl}} = 0. \tag{60}$$

Also note that due to the cylinder's isometries (translations in τ and ϕ), $\langle T_{\tau\phi} \rangle_M = 0$. Combining (58) and (60) gives

$$\langle T_{\tau\tau} \rangle = \frac{c}{24\pi} \tag{61}$$

In real time, the metric becomes:

$$ds^2_{\text{cyl}} = -dt^2 + d\phi^2, \quad t = i\tau.$$

The vacuum energy is:

$$H_{\text{vac}} = \int_{\Sigma} \langle T_{tt} \rangle_M = \int_0^{2\pi} d\phi \left(-\frac{c}{24\pi}\right) = -\frac{c}{12}.$$

You may also encounter conventions where the anomaly is defined as $\langle T^{\mu}_{\mu} \rangle = \frac{c}{12}R$, whereas we used $\langle T^{\mu}_{\mu} \rangle = \frac{c}{24\pi}R$. In the convention where $\langle T^{\mu}_{\mu} \rangle = \frac{c}{12}R$, we would need to multiply $\langle T_{zz} \rangle$ by a factor 2π. This gives

$$H_{\text{vac}} \to -\frac{\pi c}{6} \tag{62}$$

For a cylinder with radius R, this becomes $H = -\frac{\pi c}{6R}$. Nevertheless, c can be measured by the Casimir energy of a critical Hamiltonian on the cylinder.

(c) For a general coordinate transformation $z \mapsto w = f(z, \bar{z})$, $\bar{z} \mapsto \bar{w} = \bar{f}(z, \bar{z})$, the metric components transform as:

$$g_{ww} = \frac{\partial z}{\partial w}\frac{\partial \bar{z}}{\partial w}, \quad g_{\bar{w}\bar{w}} = \frac{\partial z}{\partial \bar{w}}\frac{\partial \bar{z}}{\partial \bar{w}}, \quad g_{w\bar{w}} = \frac{\partial z}{\partial w}\frac{\partial \bar{z}}{\partial \bar{w}}.$$

To remain Weyl flat, $g_{ww} = g_{\bar{w}\bar{w}} = 0$, which requires:

$$\frac{\partial \bar{z}}{\partial w} = \frac{\partial z}{\partial \bar{w}} = 0, \quad \frac{\partial z}{\partial w}, \frac{\partial \bar{z}}{\partial \bar{w}} \neq 0.$$

This implies the transformations $z = f(w), \bar{z} = \bar{f}(\bar{w})$, leading to:

$$ds^2 = \frac{\partial z}{\partial w}\frac{\partial \bar{z}}{\partial \bar{w}} dw d\bar{w}.$$

The Weyl factor is:

$$\sigma = \frac{1}{2}\ln\left(\frac{\partial w}{\partial z}\frac{\partial \bar{w}}{\partial \bar{z}}\right).$$

For this scale factor:

$$\partial_z^2\sigma - (\partial_z\sigma)^2 = \frac{1}{2}\frac{w^{(3)}(z)}{w'(z)} - \frac{3}{4}\left(\frac{w''(z)}{w'(z)}\right)^2 = \frac{1}{2}\{w, z\},$$

where $\{w, z\}$ is the Schwarzian derivative. Thus:

$$\langle T\rangle = \frac{c}{24\pi}\{w, z\}.$$

Including this anomalous term recovers the full transformation law. Putting everything together, you've proven in this exercise the appearance of the central charge c in three contexts: the Weyl anomaly, the ground state energy on the cylinder, and the conformal transformation of the stress-tensor (which implies the form of its OPE will involve c).

4. Coordinates and Isometries of AdS

(a) We will take X^0 and X^1 to be the timelike coordinates in the embedding spacetime. Global coordinates:

$$X^0 = \cosh(\rho)\cos(t), \tag{63}$$
$$X^1 = \cosh(\rho)\sin(t), \tag{64}$$
$$X^i = \sinh(\rho)\hat{n}^i, \quad (\hat{n}^i \text{ parameterizes } S^{D-2} \text{ with } \sum_{i=2}^{D+1}(\hat{n}^i)^2 = 1). \tag{65}$$

Global to Global 2:

$$r = \sinh(\rho) \tag{66}$$

Conformal coordinates:

$$X^0 = \frac{\cos(t)}{\cos(\theta)}, \tag{67}$$
$$X^1 = \frac{\sin(t)}{\cos(\theta)}, \tag{68}$$
$$X^i = \tan(\theta)\hat{n}^i. \tag{69}$$

Note that we require $\theta \in [0, \pi/2)$.
Poincaré Patch Coordinates:

$$X^0 = \frac{1}{2z}\left(1 + z^2 + \delta_{ij}x^i x^j - t^2\right), \tag{70}$$
$$X^1 = \frac{t}{z}, \tag{71}$$
$$X^i = \frac{x^i}{z}, \tag{72}$$
$$X^D = \frac{1}{2z}\left(1 - z^2 - \delta_{ij}x^i x^j + t^2\right). \tag{73}$$

Cosmological coordinates:

$$X^0 = \sin(T), \tag{74}$$
$$X^1 = \cos(T)\cosh(r), \tag{75}$$
$$X^i = \cos(T)\sinh(r)\hat{n}^i. \tag{76}$$

(b) Embedding space: $SO(2, D-1)$.
Global: time translations and rotations $SO(D-1)$.
Conformal: time translations and rotations $SO(D-1)$.
Poincaré: Poincaré group $ISO(1, D-2)$ and dilatations $z \to \lambda z, t \to \lambda t, x^i \to \lambda x^i$.
Cosmological: Hyperboloid group $SO(1, D-1)$.

(c) To find all the Killing vectors (KV) of AdS-Poincaré, our strategy is to locate the conformal Killing vectors (CKV) first, and then pick the KV from them.

Before analysing the details, we can first investigate the nature of the isometry group of AdS. We can see AdS_D as a hyperboloid embedded in $\mathbb{R}^{2,D-1}$, so it inherits the isometry group $SO(2, D-1)$.

$$x^2 + y^2 + z^2 + \cdots - t^2 - u^2 = R^2_{\text{AdS}}$$

The $(D-1, 1)$-dimension Poincaré patch of AdS has metric:

$$ds^2 = \frac{dz^2 + \eta_{ij}dx^i dx^j}{z^2} = \frac{1}{z^2}ds^2_{\text{flat}},$$

which is in the same conformal class as Minkowski, therefore, they have the same set of CKVs. The CKVs of Minkowski correspond to the $SO(2, D)$ algebra. For $A = 0, \cdots, D-1$, we have CKVs:

$$\begin{aligned} P_A &= \frac{\partial}{\partial x^A}, \\ K_A &= 2x^A\left(x \cdot \frac{\partial}{\partial x}\right) - x^2\frac{\partial}{\partial x^A}, \\ D &= x \cdot \frac{\partial}{\partial x}, \\ M_{AB} &= x^A\frac{\partial}{\partial x^B} - x^B\frac{\partial}{\partial x^A}. \end{aligned}$$

Which of these are isometries of AdS-Poincaré? Or, instead, we can first pick the ones which aren't. P_z, M_{zi}, K_z are not (for $i = 0, \cdots, D-2, z = x^{D-1}$). Thus the KVs of AdS-Poincaré are:

$$\begin{aligned}
P_i &= \frac{\partial}{\partial x^i}, \\
K_i &= 2x^i \left(z\frac{\partial}{\partial z} + x^j \frac{\partial}{\partial x^j} \right) - \left(z^2 + \eta_{jk} x^j x^k \right) \frac{\partial}{\partial x^i}, \\
D &= z\frac{\partial}{\partial z} + x^i \frac{\partial}{\partial x^i}, \\
M_{ij} &= x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}.
\end{aligned}$$

At the boundary $z = 0$, these become:

$$\begin{aligned}
P_i &= \partial_i, \\
K_i &= 2x^i (x \cdot \partial) - x^2 \partial_i, \\
D &= x \cdot \partial, \\
M_{ij} &= x^i \partial_j - x^j \partial_i.
\end{aligned}$$

These are exactly the CKVs of $(D-1)$-dimensional Minkowski spacetime, which generates the group $SO(2, D-1)$.